May 12, 2024 Math 9

Complex numbers wrapup.

Exercise. Prove that,

$$\frac{1}{1+x^2} = \frac{1}{2}\left(\frac{1}{1-ix} + \frac{1}{1+ix}\right)$$

Cardano-Tartaglia formula.

Equations involving cubic polynomial are called cubic equations. Roots of a general cubic polynomial are the solutions of an equation,

$ax^3 + bx^2 + cx + d = 0, a \neq 0$ or

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0, x^3 + Px^2 + Qx + R = 0$

Using the substitution, $x = y - \frac{b}{3a} = y - \frac{P}{3}$, this can be simplified to a reduced form, which is also called the depressed cubic equation,

$y^3 + py + q = 0$. Here $p = \frac{3ac-b^2}{3a^2}$, $q = \frac{2b^3-9abc+27a^2d}{27a^3}$.

Gerolamo Cardano published a closed formula for the solution of this equation, known as Cardano formula, in his book Ars Magna in 1545 (although a closed formula for the roots of a depressed cubic equation was first obtained 6 years earlier by Nicolo Tartaglia, who communicated his results to Cardano),

$$y = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}.$$

Derivation of Cardano is somewhat esoteric. He uses an heuristic substitution, which was later simplified by Vieta,

$$y = u - \frac{p}{3u}$$

which transforms the original equation, $y^3 + py + q = 0$, to

$$u^3 - \frac{p^3}{27u^3} - up + \frac{p^2}{3u} + p\left(u - \frac{p}{3u}\right) + q = u^3 - \frac{p^3}{27u^3} + q = 0.$$

This is a quadratic equation in $t = u^3$, $t^2 + qt - \frac{p^3}{27} = 0$. Its roots are,

$$t_{1,2} = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}$$

Cardano noticed that this equation does not always have real roots, and therefore a need arises to deal with complex numbers because we know that a cubic equation must have at least one real root. However, he did not know how to deal with this. Nevertheless, if $t_{1,2}$ are real, there is a real u, which is given by a root-three of t,

$$u = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

While there are two possible choices of $\pm$ sign in the above, they both lead to the same real $y = u - \frac{p}{3u}$, because,

$$\frac{p}{3\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \frac{p\sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3\left(\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right)\left(\sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right)} = \frac{p\sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3\left(\sqrt[3]{\left(-\frac{q}{2}\right)^2 - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right)} = \frac{p\sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3\left(\sqrt[3]{-\frac{p^3}{27}}\right)},$$

so

$$y = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This is another, equivalent expression for the Cardano formula. Note, that in the case considered by Cardano, where the square root is real, this gives a single real solution, $y = y_0$. However, this solution is obtained by extracting a root-3 of a real number, and therefore in the field of complex numbers there are three solutions, corresponding to three different roots-3 of unity. If we denote u to be the real root-3, we can write the complex solutions for y as,

$$y = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}} = u\sqrt[3]{1} - \frac{p}{3u\sqrt[3]{1}} = u\sqrt[3]{1} - \frac{p\left(\sqrt[3]{1}\right)^2}{3u},$$

where the $\sqrt[3]{1}$ has three complex values, thus identifying three complex solutions. Or, equivalently,

$$y = \sqrt[3]{1}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right) + \frac{1}{\sqrt[3]{1}}\left(\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\right).$$

This situation can be exemplified by considering the case $p = 0$, where in addition to the real root, $y_0 = \sqrt[3]{-q}$, there are also two imaginary roots,

$$y_{0,1,2} = y_0\sqrt[3]{1} = \left\{y_0, y_0\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right), y_0\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right\}$$

If, on the other hand, $\frac{q^2}{4}+\frac{p^3}{27} < 0$ and the square root in the discriminant of the quadratic equation for $t = u^3$ is imaginary, then p must be negative, and,

$$\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} = i\sqrt{\left|\frac{q^2}{4}+\frac{p^3}{27}\right|} = i\sqrt{\frac{|p|^3}{27}-\frac{q^2}{4}}, \left|\sqrt[3]{-\frac{q}{2}\pm i\sqrt{\frac{|p|^3}{27}-\frac{q^2}{4}}}\right|^2 =$$

$$\left|\sqrt[3]{\left(\frac{q}{2}\right)^2+\frac{|p|^3}{27}-\frac{q^2}{4}}\right| = \frac{|p|}{3}.$$

Consequently,

$$\frac{p}{3\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}} = \frac{p\sqrt[3]{-\frac{q}{2}\mp i\sqrt{\frac{|p|^3}{27}-\frac{q^2}{4}}}}{3\left|\sqrt[3]{-\frac{q}{2}\pm i\sqrt{\frac{|p|^3}{27}-\frac{q^2}{4}}}\right|^2} = \frac{p\sqrt[3]{-\frac{q}{2}\mp i\sqrt{\frac{|p|^3}{27}-\frac{q^2}{4}}}}{3\frac{|p|}{3}} = -\sqrt[3]{-\frac{q}{2}\mp i\sqrt{\frac{|p|^3}{27}-\frac{q^2}{4}}},$$

and the equation has three real roots,

$$y = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \sqrt[3]{-\frac{q}{2} + i\sqrt{\frac{p^3}{27} - \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - i\sqrt{\frac{p^3}{27} - \frac{q^2}{4}}},$$

which are given by the three different values of root-3 in the above expression. The roots are real because the expression is a sum of a complex number and its complex conjugate, which is always real. Note, that one has to be careful in selecting which root-3 to use in the above expression. Indeed, $y = u - \frac{p}{3u}$ has only one root-3, that for u, so we have to choose the same value for root-3 in both terms on the right hand side in the Cardano formula above. Both terms are derived from u and they are chosen such that the two above terms are complex conjugate.

Let us consider a special case, $q = 0$. In this case the Cardano formula yields,

$$y_{0,1,2} = \sqrt[3]{\left(\sqrt{\frac{p}{3}}\right)^3} - \frac{p}{3\sqrt[3]{\left(\sqrt{\frac{p}{3}}\right)^3}} = \sqrt{\frac{p}{3}}\sqrt[3]{1} - \sqrt{\frac{p}{3}}\frac{1}{\sqrt[3]{1}},$$

where the same choice of root-3 in both terms is required. Or, equivalently,

$$y_{0,1,2} = \sqrt{\frac{p}{3}}\sqrt[3]{1} - \sqrt{\frac{p}{3}}\overline{\sqrt[3]{1}}\text{, because } \left|\sqrt[3]{1}\right| = 1\text{, and therefore, } \frac{1}{\sqrt[3]{1}} = \overline{\sqrt[3]{1}}.$$

In the case $p \geq 0$, we thus obtain, $y_{0,1,2} = \{0, i\sqrt{p}, -i\sqrt{p}\}$. If $p < 0$, the roots are real, $y_{0,1,2} = \{0, \sqrt{|p|}, -\sqrt{|p|}\}$.

Trigonometric substitution for cubic equation.

More consistent derivation of the Cardano formula was given later by Lagrange. Perhaps, the best one is achieved by using a trigonometric substitution, $y = v\cos\theta$, which leads to the equation,

$$v^3\cos^3\theta + pv\cos\theta + q = 0.$$

Choosing $v = 2\sqrt{-\frac{p}{3}}$, the equation is reduced to,

$$4\cos^3\theta - 3\cos\theta - \frac{3q}{2p\sqrt{-\frac{p}{3}}} = 0$$

or,

$$\cos(3\theta) = \frac{3q}{2p\sqrt{-\frac{p}{3}}}$$

For more information on cubic equations, see http://en.wikipedia.org/wiki/Cubic_function. The only other polynomial equation that is solvable in radicals is the quartic equation, which has been solved by Cardano's student, **Ludovico Ferrari** in 1540. The solution is known as Ferrari formula, and is even more cumbersome than that of Cardano. In fact, it utilizes the latter. It was published by Cardano in his book Ars Magna together with the cubic formula in 1545 (http://en.wikipedia.org/wiki/Quartic_equation).