Homework for May 12, 2024.

Algebra/Geometry. Complex numbers.

Please, complete the previous homework assignments. Review the classwork handout on complex numbers and complete the exercises. Solve the following problems.

Problems.

1. Find all complex numbers z such that:
   a. $z^2 = -i$
   b. $z^2 = -2 + 2i\sqrt{3}$
   c. $z^3 = i$

   Hint: write and solve equations for a, b in $z = a + bi$.

2. On the complex plane, plot all fifth order roots of 1 and all fifth order roots of -1.

3.
   a. Find all roots of the polynomial $z + z^2 + z^3 + \cdots + z^n$
   b. Without doing the long division, show that $1 + z + z^2 + \cdots + z^9$ is divisible by $1 + z + z^2 + z^3 + z^4$.

4. Find the roots of the following cubic equations by heuristic guess-and-check factorization,

   a. $z^3 - 7z + 6 = 0$
   b. $z^3 - 21z - 20 = 0$
   c. $z^3 - 3z = 0$
   d. $z^3 + 3z = 0$
   e. $z^3 - \frac{3}{4}z + \frac{1}{4} = 0$

5. Which transformation of the complex plane is defined by:
    a. $z \to iz$
    b. $z \to \left(\frac{1-i}{\sqrt{2}}\right) z$
    c. $z \to (1 + i\sqrt{3})z$
    d. $z \to \frac{z}{1+i}$
    e. $z \to \frac{z+\bar{z}}{2}$
    f. $z \to 1 - 2i + z$
    g. $z \to \frac{z}{|z|}$
    h. $z \to i\bar{z}$
    i. $z \to -\bar{z}$

6. Find the sum of the following trigonometric series using de Moivre formula:

$$S_1 = \cos x + \cos 2x + \cdots + \cos nx = ?$$

$$S_2 = \sin x + \sin 2x + \cdots + \sin nx = ?$$

Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems; skip those you already solved.

Problems.

1. Write Vieta formulae for the cubic equation, $x^3 + Px^2 + Qx + R = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of P, Q and R,

   a. $(x_1 + x_2 + x_3)^2$
   b. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   c. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   d. $(x_1 + x_2 + x_3)^3$

2. The three real numbers x, y, z, satisfy the equations:

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

   a. Find a cubic polynomial whose roots are x, y, z
   b. Find x, y, z

3. Find two numbers u, v such that:

$$u + v = 6$$

$$uv = 13$$

4. Find three numbers, a, b, c, such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

5. Find all real roots of the following polynomial and factor it.

   a. $x^8 + x^4 + 1$
   b. $x^4 - x^3 + 5x^2 - x - 6$
   c. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$

6. Perform the long division, finding the quotient and the remainder, on the following polynomials.

   a. $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
   b. $(x^3 - 3x^2 + 4) \div (x^2 - 1)$